Quantum Optics I

Atom–Field Interaction SEMICLASSICAL THEORY

Ref. Sally QO. ch 5 26.

Semiclassical approach: field classical $\vec{E}(r,t)$ Atom/electron $\vec{\psi}(\vec{r})$
Fully quantum-mech interaction $\hat{E}(r,t)$ " $\hat{\vec{\psi}}(\vec{r})$

Schrödinger equation:

$$\hat{H}_0 \psi_j = E_j \psi_j = \left(-\frac{\hbar^2 \nabla^2}{2m} + V(r)\right)\psi_i(\vec{r}) \qquad \text{for e.g. electron}$$

Interaction of electron with a field: (Homework)

$$\boxed{H = \frac{1}{2m}(\underbrace{\vec{p} - e\vec{A}}_{A:\text{ vector potential}})^2 + \underbrace{eU(r,t)}_{\text{static fields}} \quad (U=0)\text{ here.}}$$

$$\vec{E} = \partial_t \vec{A} \qquad \vec{B} = \vec{\nabla}\times\vec{A} \qquad \text{since } |\psi(r,t)|^2 = P \text{ invariant!}$$

Hamiltonian can be "derived" from local (phase) gauge invariance
ie $\boxed{\psi(r,t) = e^{i\chi(r,t)}\psi(r,t)}$. $\overset{A+}{A'} \propto \frac{\hbar}{e}\nabla\chi$ $\overset{U+}{U'} \propto -\frac{\hbar}{e}\partial_t\chi$ → Photon emerges from local gauge invariance!

First note that electrons are fermions. (anti-commutator relations eg $(a_i^\dagger)^2|0\rangle = |0\rangle$).

DIPOLE APPROXIMATION AND $\vec{r}\cdot\vec{E}$ HAMILTONIAN

$$\left[-\frac{\hbar^2}{2m}\left(\vec{\nabla} - \frac{ie\,A(r,t)}{\hbar}\right)^2 + V(r)\right]\psi(\vec{r}) = +\partial_t\psi(r)\cdot i\hbar$$

Note that $|\psi(\vec{r})|^2$ extends typically of a_0 (Bohr radius)

and $a_0 \ll \lambda$. Therefore we have.

$$A(r+r_0) = A(t)\,e^{i\vec{k}(\vec{r}+\vec{r}_0)} \simeq A(t)\,e^{i\vec{k}\vec{r}_0}\,(1+\underbrace{i\vec{k}\vec{r}}_{\text{neglect.}})$$
$$\simeq A(t)\,e^{i\vec{k}\vec{r}_0}.$$

Inserting in Coulomb Gauge:

$$V(r) = 0 \quad \text{and} \quad \vec{\nabla}\cdot\vec{A} = 0$$

yields:

$$-\frac{\hbar^2}{2m}\left(\vec{\nabla} - \frac{ie\,A(r_0,t)}{\hbar}\right)\psi(\vec{r}) = +i\hbar\partial_t\psi(\vec{r})$$

Introduce new wave function: Note $|\psi(r)|^2 = Pr(r) = |\Phi(r,t)|^2$ is INVARIANT.

$$\boxed{\psi(\vec{r}) \equiv \exp\left(\frac{ie}{\hbar}\vec{A}(\vec{r}_0,t)\cdot\vec{r}\right)\Phi(\vec{r},t)}$$

→ Göppert-Mayer Transformation. Makes $A' = 0$

Equivalent to gauge. $\chi \equiv -e(\vec{A}(r,t)/\hbar)\cdot\vec{r}$ in wave-fct.

Inserting:

(neglecting A^2).

$$i\hbar\left[\left(\frac{ie}{\hbar}\right)\underbrace{\partial_t\vec{A}(r_0,t)\cdot\vec{r}}_{\vec{E}\cdot\vec{r}}\Phi(\vec{r},t) + \dot{\Phi}(r,t)\right]e^{\frac{ie}{\hbar}\vec{A}\cdot\vec{r}} = e^{\frac{ie}{\hbar}\vec{A}\cdot\vec{r}}\left[\frac{p^2}{2m}+V\right]\Phi(r,t)$$

ie Note that:

$$i\hbar\partial_t\left(\Phi(r)e^{-\vec{A}(r_0,t)\cdot\vec{r}}\right) = i\hbar\dot{\Phi}e^{-\vec{A}(r_0,t)\cdot\vec{r}} + i\hbar\Phi\frac{dA(r_0,t)}{dt}\cdot\vec{r}\cdot c$$

1st Term cancels

$$\Rightarrow \left(-e\vec{E}\cdot\vec{r}\right)\Phi + i\hbar\partial_t\Phi(r,t) = \overbrace{\left(\frac{p^2}{2m}+V\right)}^{H_0}\Phi$$

$$\boxed{\text{i.e. } i\hbar\partial_t\Phi(r,t) = [H_0 - \underbrace{e\vec{r}\cdot\vec{E}(\vec{r}_0,t)}_{\text{Dipole-Hamiltonian}}]\Phi(\vec{r},t)} \wedge\ p \equiv -i\nabla\hbar$$

$$\hat{H}_{int} = -e\vec{E}(r_0,t)\cdot\vec{r}$$

Note: $\left|\begin{array}{l} A' \Rightarrow -A(r_0) + A(r) \\ V' \Rightarrow U + r\cdot\vec{A} \end{array}\right|$ TRANSFORMATION.

Quantum mechanically (operator form)

$$\boxed{\hat{H}_{int} = -e\hat{\vec{E}}(\hat{\vec{r}}_0,t)\,\underbrace{\hat{\vec{r}}}_{\text{operator}}}$$

$\hat{r} \to r_0$ in Dipole approximation.

$\hat{\vec{r}}\,|\vec{r}\rangle = \vec{r}\,|\vec{r}\rangle$

position eigenstate ie $\delta(r-r_0)$

Alternative derivation: $p\cdot A$ HAMILTONIAN

$$H = \frac{1}{2m}(\underbrace{-i\hbar\vec{\nabla}}_{p} - e\vec{A}(r_0,t))^2 + V(r)$$

Hamiltonian with $A(\vec{r},t) \simeq A_0(r_0,t)$ DIPOLE APPROX.

$$= \Big(\underbrace{\frac{e}{m}(-\vec{p}\cdot\vec{A})}_{H_{int}} + \frac{e^2}{2m}\vec{A}(r_0,t)^2 \underbrace{-\frac{1}{2m}(-i\hbar\nabla)^2}_{H_0}\Big) + V(r)$$

Thus: $$\boxed{[H_0 - \frac{e}{m}\vec{p}\cdot\vec{A} + \underbrace{\frac{e^2}{m^2}\vec{A}^2(r_0,t)}]\psi(r,t) = +i\hbar\partial_t\psi(r,t)}$$

neglected, ie small compared to $\vec{p}\cdot\vec{A}$

vibrational kinetic energy of free particle.

Note that since

$\cancel{\vec{B} \leq \vec{\nabla}\times\vec{A}}$ ~~ie B ∝ k×A~~

$\rightarrow \cancel{\frac{e^2}{m^2}A^2 = \frac{e}{m}\frac{A^2}{p} = \frac{e}{m}}$ $\frac{\cancel{e}}{m}Ap$

$$\frac{H_{12}}{H_1} = \frac{e^2A^2/m}{e\,Ap/m} \simeq \frac{e^2E^2}{2m\omega^2} \ll 1$$

for $A = \frac{E}{\omega}$

ie for low intensity fields:

$$\boxed{H = H_0 - \frac{e}{m}\vec{p}\cdot\vec{A}(r_0,t)}$$ $\vec{p}\cdot\vec{A}$ approximation.

$\Rightarrow$ There are two equivalent formulations $\vec{p}\cdot\vec{A}$ or $\vec{r}\cdot\vec{E}$.

Quantum Optics I

Q-Electrodynamics (semiclassical). p·A interaction Hamiltonian and DIPOLE APPROXIMATION.

$$\boxed{\hat{H}_{int} = -\hat{p}\cdot A(\hat{r},t)\frac{e}{m}}$$

$|\psi\rangle = |\psi\rangle_{electron}$ (semiclassical) $|\psi\rangle = a_j$

$$\langle\psi|\hat{H}_{int}|\psi\rangle = \langle\Phi_1|\hat{H}_{int}|\Phi_2\rangle \quad (\text{eigenstate } |\Phi_j\rangle = \{|1\rangle, |2\rangle\})$$

$$= -\frac{e}{m}\underbrace{\int \Phi_1^*(\vec{r})\,\underset{\substack{\downarrow\\ -i\hbar\vec{\nabla}_x}}{\hat{p}}\,\Phi_2(\vec{r})\,d^3\vec{r}}_{\substack{\text{Spatial extend}\\ \to \text{ Bohr radius } a_0}}\cdot\underbrace{\vec{u}_k(\vec{r})\sqrt{\frac{1}{2\hbar\omega\varepsilon_0}}}_{\substack{A(\vec{r},t)\,:\,\text{Vector potential}\\ \text{(classical)}\\ \text{spatial extend} \to \text{ wavelength } \lambda}}$$

$\to \lambda \gg a_0$

and $u_k(\vec{r}) = \frac{1}{\sqrt{V}}e^{i\vec{k}\cdot\vec{r}}$: plane wave

$\to$ assume that $e^{i\vec{k}\vec{r}} \simeq e^{i\vec{k}\vec{r}_0}(1 + i\vec{k}\,\delta\vec{r} + \ldots) \approx e^{i\vec{k}\vec{r}_0}$

Hence:

$$\langle\Phi_1|\hat{H}_{int}|\Phi_2\rangle = -\frac{e}{m}\sqrt{\frac{1}{2\hbar\omega_0\varepsilon_0}}\,e^{i\vec{k}\vec{r}_0}\cdot\int\Phi_1^*(\vec{r})\cdot\underset{\substack{\downarrow\\ -i\hbar\vec{\partial}}}{\hat{p}}\,\Phi_2(\vec{r})\,d^3r$$

can be rewritten as (cf. Milburn):

$$\int\Phi_1^*\hat{p}\,\Phi_2\,d^3r = \frac{i\cdot m}{\hbar}\int\Phi_1^*[H_{electron}, \hat{x}]\,\Phi_2(\vec{r})\,d^3r \qquad H = \sum \hat{a}_j^\dagger\hat{a}_j E_j$$

$$= \frac{im}{\hbar}(E_2 - E_1)\cdot\int\Phi_1^*(\vec{r})\,x\,\Phi_2(\vec{r})\,d^3r$$

$$\boxed{\langle 2|\hat{\Phi}_{ep}|1\rangle = \frac{im}{\hbar}(\hbar\omega_{12})\int\Phi_1^*(\vec{r})\cdot\vec{r}\,\Phi_2(\vec{r})\,d^3r}$$

$\to$ similar to $-eE$ Dipole Coupling term again

Some consequences:

Dipole moment vanishes for orbitals of ~~both~~ same parity.

Quantum Optics I

Interaction of field with 2 level system: Semiclassical analysis.

$\psi(x) = \sum_j \hat{a}_j \psi_j(x)$

E.g. consider first two Energy levels $H_0 = \sum_{j=1}^{2} \hat{a}_j^+ a_j E_j$

E_1 ψ_1 a_1 $j=2$
E_0 ψ_0 a_0 $j=1$

$|\psi|^2$

E_1
E_0
(potential well)

for a ground state $\boxed{\begin{array}{l} a_j|0\rangle = 0 \\ a_j^+|0\rangle = |1\rangle \end{array}}$ (with engy E_j)

For Fermions we have $\boxed{(\hat{a}_j^+)^2|\Phi\rangle = 0}$ ie no two electrons in the same level.

$|\{j\}\rangle = \underbrace{\hat{a}_{j1}^+ \dots \hat{a}_j^+ |0\rangle}_{\text{creates } n\text{-electron engy state}}$ ie ~~$\psi_1\psi_2\dots\psi_n$ state~~

(For single electron systems simply satisfied)

Electrons obey anti-commutation relations:

$$\boxed{\hat{a}_j^+\hat{a}_k^+ + \hat{a}_k^+\hat{a}_j^+ = \hat{a}_j\hat{a}_k + \hat{a}_k\hat{a}_j = 0 \quad \text{and} \quad \hat{a}_j^+\hat{a}_k + \hat{a}_k\hat{a}_j^+ = \delta_{jk}}$$

In Quantum Optics we are concerned with single electron states and thus have:

$$|j\rangle = a_j^+|0\rangle \quad \text{ie 1-electron state.} \quad ((a^+)^2|0\rangle = 0)$$

Pseudo-spin operators:

$$\boxed{\begin{array}{l} a_2^+ a_1 = \sigma^+ \\ a_1 a_2^+ = \sigma^- \\ \frac{1}{2}(a_2^+ a_2^+ - a_1^+ a_1^+) = \sigma_z \end{array}}$$

$\{\sigma^+, \sigma^-, \sigma_z\}$ obey same anti-commutation relations

$\boxed{\hat{a}_1^+\hat{a}_1 + \hat{a}_2^+\hat{a}_2 = 1 \text{ in a 2-level system}}$

Thus the operators can be presented by projectors equally in $\{|1\rangle, |2\rangle\}$ Basis:

$$\boxed{\begin{array}{lll} \hat{\sigma}^+ = |2\rangle\langle 1| & \hat{\sigma}^- = |1\rangle\langle 2| & \hat{\sigma}_z = \frac{1}{2}(-|1\rangle\langle 1| + |2\rangle\langle 2|) \\ H_0|1\rangle = -\frac{\hbar\omega_0}{2} & H_0|2\rangle = +\frac{\hbar\omega_0}{2} & H_0 = \hbar\omega\hat{\sigma}_z \\ S_{22} = \langle\hat{\sigma}^+\hat{\sigma}^-\rangle & \sigma_{11} = \langle\hat{\sigma}^-\hat{\sigma}^+\rangle & S_{22} - S_{11} = 2\langle\sigma_z\rangle \end{array}}$$

Next apply formalism of Pseudo spin operator to ~~free~~ single electron interacting with field in $-e\vec{r}\cdot\vec{E}$ approximation. Semiclassical

$-\vec{E}(\hat{r}, t)\cdot\hat{r}\, e$
operator → acts on $\psi(\vec{r})$ not $\vec{E}$ Hilbert space

Added Notes

$\psi(x) = \sum \hat{a}_j \psi_j(x)$

$\hat{a}_j$: operators in expansion.
ψ: field operator

$\hat{H}_0 = \sum \hat{a}_j^+ \hat{a}_j E_j$ Energy of level j. For electron field operators $\psi(\vec{x})$

$\psi(x)\psi(x') = -\psi(x')\psi(x)$ anti-symmetric wave function
$\psi(x)\psi^+(x') + \psi^+(x')\psi(x) = \delta(x - x')$

Lecture 5: Semiclassical Atom Field interaction

Consider 2-level system $\{|1\rangle, |2\rangle\}$ $\quad \omega_{12}$: $|2\rangle$, $|1\rangle$

$|\psi\rangle = c_1(t)|1\rangle + c_2(t)|2\rangle$ ie Probability amplitudes in Schrödinger picture

$\frac{d}{dt}|\psi\rangle = -\frac{i}{\hbar}\hat{H}|\psi\rangle$ $\quad$ Transition freq $\omega_{12} = \omega_2 - \omega_1$

$H_0 = \hbar\omega_1 |1\rangle\langle 1| + \hbar\omega_2 |2\rangle\langle 2|$ $\quad (\hat{H}_0 = \hat{\sigma}_z \hbar\omega_{12})$

$\hat{H}_{int} = e\hat{x}E(t) = e\hat{x}\mathcal{E}\cos(\omega_L t)$

Evaluate coupling terms eg $\langle i|\hat{H}_{int}|j\rangle$. Note that the terms $\langle i|\hat{x}|i\rangle$ vanish due to parity ie $\int |\Phi_i|^2 \vec{r}\, d^3r = 0$

Hence only 2-non zero terms $\langle 1|e\hat{x}E(t,x_0)|2\rangle$ and $\langle 2|e\hat{x}E(t,x_0)|1\rangle$

$$\boxed{\text{Thus: } \hat{H}_{int} = \underbrace{\left(\int \Phi_1^*(\vec{r})\, e\vec{r}\, \Phi_2(\vec{r})\, d^3r\right)}_{\langle p_{ab}\rangle_{12}} \cdot \underbrace{|1\rangle\langle 2|}_{\hat{\sigma}_-}\vec{E}(t) + \underbrace{\left(\int \Phi_2^*(\vec{r})\, e\vec{r}\, \Phi_1(\vec{r})\, d^3r\right)}_{p_{12}^*} \underbrace{|2\rangle\langle 1|}_{\hat{\sigma}_+}\vec{E}(t)}$$

Semiclassical atom field interaction

$|\psi(t)\rangle = c_1(t)|1\rangle + c_2(t)|2\rangle$

Equations of Motion $\quad \frac{d}{dt}c_1(t)\langle 1|\psi\rangle = \langle 1|-\frac{i}{\hbar}\hat{H}|\psi\rangle$ same for c_2

$$\frac{d}{dt}c_1(t) = \langle 1|\left(-\frac{i}{\hbar}\hat{H}|\psi\rangle\right) = \underbrace{-i\omega_1 c_1(t)}_{\text{eliminate by rot frame}} + i\underbrace{\frac{p_{12}}{\hbar}E(t)}_{\text{time dependent}} \cdot c_2(t)$$

$$\frac{d}{dt}c_2(t) = -i\omega_2 c_2(t) + i\frac{p_{21}^*}{\hbar}E(t)\, c_1(t) = \frac{d}{dt}\cdot\langle 2|\psi\rangle$$

First rotating frame with $c_1(t) = c_1(t)e^{+i\omega_1 t}$, $c_2(t) = c_2(t)e^{+i\omega_2 t}$ →

Note that $E(t) = \mathcal{E}\cos(\omega_L t)$ $\quad \boxed{\Omega_R \equiv \frac{|p_{12}|\mathcal{E}}{\hbar}}$ Rabi frequency

rotating frame: $\omega \equiv \omega_1 - \omega_2$

$$\dot{c}_1 = i\Omega_R/2\; c_2\left(e^{+i(\omega-\omega_L)t} + e^{+i(\omega+\omega_L)t}\right)$$
$$\dot{c}_2 = i\Omega_R/2\; c_1\left(e^{-i(\omega-\omega_L)t} + \underbrace{e^{-i(\omega+\omega_L)t}}_{\text{counter rotating term → neglect.}}\right)$$

eliminate time dependence $\tilde{c}_1 = c_1(t)e^{-i\overbrace{(\omega-\omega_L)}^{\Delta/2}t/2}$, $\tilde{c}_2 = c_2(t)e^{+i\Delta/2\cdot t}$ $\Big\}$ becomes stationary frame.

$$\Leftrightarrow \frac{d}{dt}\begin{pmatrix}\tilde{c}_1\\ \tilde{c}_2\end{pmatrix} = \begin{pmatrix}-i\Delta/2 & i\Omega_R/2\\ i\Omega_R/2 & +i\Delta/2\end{pmatrix}\begin{pmatrix}\tilde{c}_1\\ \tilde{c}_2\end{pmatrix}$$

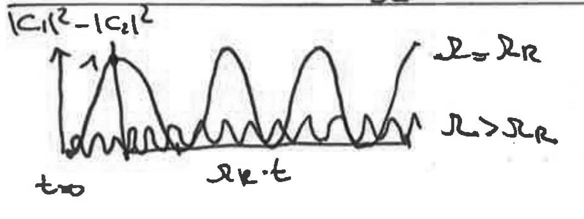
⇒ ROTATING WAVE APPROXIMATION. $\Delta \equiv \omega - \nu_L$

Solution: $\Omega \equiv \sqrt{\Omega_R^2 + (\omega-\nu_0)^2}$ $\wedge$ $\boxed{c_1 = \left\{c_1(0)\left[\cos\left(\frac{\Omega t}{2}\right) - \frac{i\Delta}{\Omega}\sin\left(\frac{\Omega t}{2}\right)\right] + i\frac{\Omega_R}{\Omega}c_2(0)\sin\left(\frac{\Omega t}{2}\right)\right\}e^{i\Delta t/2}}$

~~correspondence~~: inversion: $\Delta \equiv \omega - \omega_L$

$$|c_1|^2 - |c_2|^2 = \left(\frac{\Delta^2 - \Omega_R^2}{\Omega^2}\right)\sin^2\left(\frac{\Omega t}{2}\right) + \cos^2\left(\frac{\Omega t}{2}\right)$$

ie Rabi oscillations occur



note: for $\Delta \gg \Omega_R$ $\quad$ Thus $\cos^2 + \sin^2 = 1$ → no oscillations!

$$\frac{\Delta^2 - \Omega_R^2}{\Omega^2} = \frac{\Delta^2 - \Omega_R^2}{\Delta^2 + \Omega_R^2} \to 1 \quad (-1 \text{ for } \Delta = 0)$$

Quantum Optics

Dipole operator. $\langle p\rangle = \langle\psi| er |\psi\rangle$

$$P(t) = 2\,\mathrm{Re}\left\{ \frac{i\Omega_K}{\Omega} \, \wp_{12} \left[\cos\left(\frac{\Omega t}{2}\right) + \frac{i\Delta}{\Omega}\sin\left(\frac{\Omega t}{2}\right)\right] \sin\left(\frac{\Omega t}{2}\right) e^{i\Phi} e^{i\nu t} \right\}$$

Full quantized model: both Electrical Field and Atom are treated quantum mechanically

$$\hat{E}(t) = \underbrace{\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}}_{\text{(vacuum Field)}} \left(\hat{a}\, e^{-i\vec{k}\vec{r}} + \hat{a}^+ e^{+i\vec{k}\vec{r}}\right) \vec{u}_k$$

we obtain Atom Field interaction Hamiltonian:

$$\boxed{\hat{H} = \underbrace{\hbar\omega_k \hat{a}_k^+ \hat{a}_k}_{\text{Field}} + \underbrace{\hbar\omega_{12}\hat{\sigma}_z}_{\text{TLS}} + \sum\left\{ g_k \hat{\sigma}^+ (\hat{a}_k + \hat{a}_k^+) + g_k \hat{\sigma}^- (\hat{a}_k + \hat{a}_k^+)\right.}$$

$g_k = -\frac{p_{12} u_k \mathcal{E}_k}{\hbar}$ (Rabi frequency ie coupling strength)

Notice that $\sigma^+ a_k^+ |n 1\rangle = |n+1, 2\rangle$ ie causes change of total system energy by $2\hbar\omega$ → energy non-conserving. In dynamics causes rotation with $\propto \pm 2i\omega$ for $\{\hat{\sigma}^+ \hat{a}_k^+, \hat{\sigma}^- \hat{a}_k\}$

Eigenstates of the Hamiltonian in R.W.A.: $|n\rangle \otimes \{|1\rangle, |2\rangle\}$

$$\left|\begin{aligned} \frac{d}{dt} c_{n,2} &= -i\Omega_n \cdot c_{n,2} \\ \frac{d}{dt} c_{n+1,1} &= -i\Omega_n \cdot c_{n+1,1} \end{aligned}\right|$$

And $\Omega_n = g\sqrt{n+1}$ Rabi-frequency.

<u>NOTICE:</u>

$$\langle n 2 | e\hat{\vec{r}}\hat{\vec{E}} | n+1, 1\rangle = \underbrace{(e\langle 2|r|1\rangle)}_{\text{dipole matrix element}} \cdot \underbrace{\langle n|\hat{E}|n+1\rangle}_{\substack{\text{RWA } a_k^+ \text{ term only} \\ (\sqrt{n+1})}}$$